

EXCEPTIONAL SETS IN WARING'S PROBLEM: TWO SQUARES AND s BIQUADRATES

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ABSTRACT. Let $R_s(n)$ denote the number of representations of the positive number n as the sum of two squares and s biquadrates. When $s = 3$ or 4, it is established that the anticipated asymptotic formula for $R_s(n)$ holds for all $n \leq X$ with at most $O(X^{(9-2s)/8+\epsilon})$ exceptions.

1. INTRODUCTION

Waring's problem for sums of mixed powers involving one or two squares has been widely investigated. In 1987-1988, Brüdern [1, 2] considered the representation of n in the form

$$n = x_1^2 + x_2^2 + y_1^{k_1} + \cdots + y_s^{k_s},$$

with $k_1^{-1} + \cdots + k_s^{-1} > 1$. Linnik [8] and Hooley [6] investigated sums of two squares and three cubes. In 2002, Wooley [11] investigated the exceptional set related to the asymptotic formula in Waring's problem involving one square and five cubes. Recently, Brüdern and Kawada [3] established the asymptotic formula for the number of representations of the positive number n as the sum of one square and seventeen fifth powers.

Let $R_s(n)$ denote the number of representations of the positive number n as the sum of two squares and s biquadrates. Very recently, subject to the truth of the Generalised Riemann Hypothesis and the Elliott-Halberstam Conjecture, Friedlander and Wooley [4] established that $R_3(n) > 0$ for all large n under certain congruence conditions. They also showed that if one is prepared to permit a small exceptional set of natural numbers n , then the anticipated asymptotic formula for $R_s(n)$ can be obtained. To state their results precisely, we introduce some notations. We define

$$(1.1) \quad \mathfrak{S}_s(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^q q^{-2-s} S_2(q, a)^2 S_4(q, a)^s e(-na/q),$$

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where the Gauss sum $S_k(q, a)$ is defined as

$$(1.2) \quad S_k(q, a) = \sum_{r=1}^q e(ar^k/q).$$

As in [4], we refer a function $\psi(t)$ as being a *sedately increasing function* when $\psi(t)$ is a function of a positive variable t , increasing monotonically to infinity, and satisfying the condition that when t is large, one has $\psi(t) = O(t^\delta)$ for a positive number δ sufficiently small in the ambient context. Then we introduce $E_s(X, \psi)$ to denote the number of integers n with $1 \leq n \leq X$ such that

$$(1.3) \quad \left| R_s(n) - c_s \Gamma\left(\frac{5}{4}\right)^4 \mathfrak{S}_s(n) n^{s/4} \right| > n^{s/4} \psi(n)^{-1},$$

where $c_3 = \frac{2}{3}\sqrt{2}$ and $c_4 = \frac{1}{4}\pi$. Friedlander and Wooley [4] established the upper bounds

$$(1.4) \quad E_3(X, \psi) \ll X^{1/2+\varepsilon} \psi(X)^2$$

and

$$(1.5) \quad E_4(X, \psi) \ll X^{1/4+\varepsilon} \psi(X)^4,$$

where $\varepsilon > 0$ is arbitrary small.

The main purpose of this note is to establish the following result.

Theorem 1.1. *Suppose that $\psi(t)$ is a sedately increasing function. Let $E_s(X, \psi)$ be defined as above. Then for each $\varepsilon > 0$, one has*

$$(1.6) \quad E_3(X, \psi) \ll X^{3/8+\varepsilon} \psi(X)^2$$

and

$$(1.7) \quad E_4(X, \psi) \ll X^{1/8+\varepsilon} \psi(X)^2,$$

where the implicit constants may depend on ε .

We establish Theorem 1.1 by means of the Hardy-Littlewood method. In order to estimate the corresponding exceptional sets effectively, we employ the method developed by Wooley [10, 11].

As usual, we write $e(z)$ for $e^{2\pi iz}$. Whenever ε appears in a statement, either implicitly or explicitly, we assert that the statement holds for each $\varepsilon > 0$. Note that the "value" of ε may consequently change from statement to statement. We assume that X is a large positive number, and $\psi(t)$ is a sedately increasing function.

2. PREPARATIONS

Throughout this section, we assume that $X/2 < n \leq X$. For $k \in \{2, 4\}$, we define the exponential sum

$$f_k(\alpha) = \sum_{1 \leq x \leq P_k} e(\alpha x^k),$$

where $P_k = X^{1/k}$. We take s to be either 3 or 4. By orthogonality, we have

$$(2.1) \quad R_s(n) = \int_0^1 f_2(\alpha)^2 f_4(\alpha)^s e(-n\alpha) d\alpha.$$

When Q is a positive number, we define $\mathfrak{M}(Q)$ to be the union of the intervals

$$\mathfrak{M}_Q(q, a) = \{\alpha : |q\alpha - a| \leq QX^{-1}\},$$

with $1 \leq a \leq q \leq Q$ and $(a, q) = 1$. Whenever $Q \leq X^{1/2}/2$, the intervals $\mathfrak{M}_Q(q, a)$ are pairwise disjoint for $1 \leq a \leq q \leq Q$ and $(a, q) = 1$. Let ν be a sufficiently small positive number, and let $R = P_4^\nu$. We take $\mathfrak{M} = \mathfrak{M}(R)$ and $\mathfrak{m} = (R/N, 1 + R/N] \setminus \mathfrak{M}$. Write

$$v_k(\beta) = \int_0^{P_k} e(\gamma^k \beta) d\gamma.$$

One has the estimate

$$v_k(\beta) \ll P_k(1 + X|\beta|)^{-1/k}.$$

For $\alpha \in \mathfrak{M}_{X^{1/2}/2}(q, a) \subseteq \mathfrak{M}(X^{1/2}/2)$, we define

$$(2.2) \quad f_k^*(\alpha) = q^{-1} S_k(q, a) v_k(\alpha - a/q).$$

It follows from Theorem 4.1 [9] that whenever $\alpha \in \mathfrak{M}_{X^{1/2}/2}(q, a)$, one has

$$(2.3) \quad f_k(\alpha) - f_k^*(\alpha) \ll q^{1/2} (1 + X|\alpha - a/q|)^{1/2} X^\varepsilon.$$

We define the multiplicative function $w_k(q)$ by taking

$$w_k(p^{uk+v}) = \begin{cases} kp^{-u-1/2}, & \text{when } u \geq 0 \text{ and } v = 1, \\ p^{-u-1}, & \text{when } u \geq 0 \text{ and } 2 \leq v \leq k. \end{cases}$$

Note that $q^{-1/2} \leq w_k(q) \ll q^{-1/k}$. Whenever $(a, q) = 1$, we have

$$q^{-1} S_k(q, a) \ll w_k(q).$$

Therefore for $\alpha = a/q + \beta \in \mathfrak{M}_{X^{1/2}/2}(q, a) \subseteq \mathfrak{M}(X^{1/2}/2)$, one has

$$(2.4) \quad f_k^*(\alpha) \ll w_k(q) P_k (1 + X|\beta|)^{-1/k} \ll P_k q^{-1/k} (1 + X|\beta|)^{-1/k}.$$

The following conclusion is (4.1) in [4].

Lemma 2.1. *One has*

$$\int_{\mathfrak{M}} f_2(\alpha)^2 f_4(\alpha)^s e(-n\alpha) d\alpha = c_s \Gamma\left(\frac{5}{4}\right)^4 \mathfrak{S}_s(n) n^{s/4} + O(n^{s/4-\kappa+\varepsilon}),$$

for a suitably small positive number κ .

The next result provides the value of the Gauss sum $S_2(q, a)$.

Lemma 2.2. *The Gauss sum $S_2(q, a)$ satisfies the following properties.*

(i) *If $(2a, q) = 1$, then*

$$S_2(q, a) = \left(\frac{a}{q}\right) S_2(q, 1).$$

Here by $\left(\frac{a}{q}\right)$ we denote the Jacobi symbol.

(ii) *If q is odd, then*

$$S_2(q, 1) = \begin{cases} q^{1/2}, & \text{if } q \equiv 1 \pmod{4}, \\ iq^{1/2}, & \text{if } q \equiv 3 \pmod{4}. \end{cases}$$

(iii) *If $(2, a) = 1$, then*

$$S_2(2^m, a) = \begin{cases} 0, & \text{if } m = 1, \\ 2^{m/2}(1 + i^a), & \text{if } m \text{ is even}, \\ 2^{(m+1)/2}e(a/8), & \text{if } m > 1 \text{ and } m \text{ is odd}. \end{cases}$$

(iv) *If $(q_1, q_2) = 1$, then*

$$S_2(q_1 q_2, a_1 q_2 + a_2 q_1) = S_2(q_1, a_1) S_2(q_2, a_2).$$

Proof. These properties can be found in Lemma 2 [5]. □

3. THE PROOF OF THEOREM 1.1

Let τ be a fixed sufficiently small positive number. Set $Y = P_4^{3/2+\tau} \psi(X)^2$. We define $\mathfrak{m}_1 = \mathfrak{m} \setminus \mathfrak{M}(X^{1/2}/2)$, $\mathfrak{m}_2 = \mathfrak{M}(X^{1/2}/2) \setminus \mathfrak{M}(Y)$, $\mathfrak{m}_3 = \mathfrak{M}(Y) \setminus \mathfrak{M}(P_4)$ and $\mathfrak{m}_4 = \mathfrak{M}(P_4) \setminus \mathfrak{M}$. Let $\eta(n)$ be sequence of complex numbers satisfying $|\eta(n)| = 1$. Let \mathcal{Z} be a subset of $\{n \in \mathbb{N} : X/2 < n \leq X\}$. We abbreviate $\text{card}(\mathcal{Z})$ to Z . Then we introduce the exponential sum $\mathcal{E}(\alpha)$ by

$$\mathcal{E}(\alpha) = \sum_{n \in \mathcal{Z}} \eta(n) e(-n\alpha).$$

For $1 \leq j \leq 4$, we define

$$(3.1) \quad \mathcal{I}_j = \int_{\mathfrak{m}_j} |f_2(\alpha)^2 f_4(\alpha)^s \mathcal{E}(\alpha)| d\alpha.$$

Lemma 3.1. *Let \mathcal{I}_1 be defined in (3.1). Then we have*

$$\mathcal{I}_1 \ll P_4^{4-\frac{1}{4}+\frac{s-3}{2}+\varepsilon} Z^{1/2} + P_4^{s-\frac{1}{4}+\varepsilon} Z.$$

Proof. For any $\alpha \in \mathfrak{m}_1$, there exist a and q with $1 \leq a \leq q \leq 2X^{1/2}$ and $(a, q) = 1$ such that $|q\alpha - a| \leq X^{-1/2}/2$. Since $\alpha \in \mathfrak{m}_1$, we conclude that $q > X^{1/2}/2$. It follows from Weyl's inequality (Lemma 2.4 [9]) that

$$f_2(\alpha) \ll P_2^{1/2+\varepsilon} \quad \text{for } \alpha \in \mathfrak{m}_1.$$

Thus we have

$$\begin{aligned} \mathcal{I}_1 &\ll P_2^{1+\varepsilon} \int_{\mathfrak{m}_1} |f_4(\alpha)^s \mathcal{E}(\alpha)| d\alpha \\ &\ll P_2^{1+\varepsilon} \left(\int_0^1 |f_4(\alpha)^6| d\alpha \right)^{\frac{1}{2}} \left(\int_0^1 |f_4(\alpha)^{2(s-3)} \mathcal{E}(\alpha)^2| d\alpha \right)^{\frac{1}{2}}. \end{aligned}$$

By Hua's inequality (Lemma 2.5 [9]) and Schwartz's inequality,

$$\int_0^1 |f_4(\alpha)^6| d\alpha \ll \left(\int_0^1 |f_4(\alpha)^4| d\alpha \right)^{1/2} \left(\int_0^1 |f_4(\alpha)^8| d\alpha \right)^{1/2} \ll P_4^{7/2+\varepsilon}.$$

When $s = 4$, one has the bound $\int_0^1 |f_4(\alpha)^{2(s-3)} \mathcal{E}(\alpha)^2| d\alpha \ll P_4 Z + P_4^\varepsilon Z^2$. Then we can conclude that

$$(3.2) \quad \mathcal{I}_1 \ll P_4^{4-\frac{1}{4}+\frac{s-3}{2}+\varepsilon} Z^{1/2} + P_4^{s-\frac{1}{4}+\varepsilon} Z.$$

Indeed when $s = 3$, the estimate (3.2) holds with $P_4^{s-\frac{1}{4}+\varepsilon} Z$ omitted. \square

Lemma 3.2. *Let \mathcal{I}_2 be given by (3.1). Then one has*

$$\mathcal{J}_2 \ll P_4^{4-\frac{1}{4}+\frac{s-3}{2}+\varepsilon} Z^{1/2} + P_4^{s-\tau/2+\varepsilon} \psi(X)^{-1} Z.$$

Proof. We introduce

$$\mathcal{J}_1 = \int_{\mathfrak{m}_2} |(f_2(\alpha) - f_2^*(\alpha))^2 f_4(\alpha)^s \mathcal{E}(\alpha)| d\alpha$$

and

$$\mathcal{J}_2 = \int_{\mathfrak{m}_2} |f_2^*(\alpha)^2 f_4(\alpha)^s \mathcal{E}(\alpha)| d\alpha.$$

Note that $|f_2(\alpha)|^2 \ll |f_2(\alpha) - f_2^*(\alpha)|^2 + |f_2^*(\alpha)|^2$, where $f_2^*(\alpha)$ is defined in (2.2). Then one has

$$(3.3) \quad \mathcal{I}_2 \ll \mathcal{J}_1 + \mathcal{J}_2.$$

In view of (2.3), we know $f_2(\alpha) - f_2^*(\alpha) \ll P_2^{1/2+\varepsilon}$ for $\alpha \in \mathfrak{m}_2$. The argument leading to (3.2) also implies

$$(3.4) \quad \mathcal{J}_1 \ll P_4^{4-\frac{1}{4}+\frac{s-3}{2}+\varepsilon} Z^{1/2} + P_4^{s-\frac{1}{4}+\varepsilon} Z.$$

One has, by Schwartz's inequality, that

$$\mathcal{J}_2 \leq \left(\int_{\mathfrak{m}_2} |f_4(\alpha)^6| d\alpha \right)^{1/2} \mathcal{J}^{1/2} \ll P_4^{7/4+\varepsilon} \mathcal{J}^{1/2},$$

where \mathcal{J} is defined as

$$\mathcal{J} = \int_{\mathfrak{m}_2} |f_2^*(\alpha)^4 f_4(\alpha)^{2(s-3)} \mathcal{E}(\alpha)^2| d\alpha.$$

In order to handle \mathcal{J} , we need the following estimate

$$(3.5) \quad \int_{\mathfrak{m}_2} |f_2^*(\alpha)^4| e(-h\alpha) d\alpha = \begin{cases} O(P_4^{4+\varepsilon} Y^{-1}), & \text{when } 0 < |h| \leq 2X, \\ O(P_4^{4+\varepsilon}), & \text{when } h = 0. \end{cases}$$

Recalling the definition of $f_2^*(\alpha)$, we conclude that

$$\begin{aligned} & \int_{\mathfrak{m}_2} |f_2^*(\alpha)^4| e(-h\alpha) d\alpha \\ &= \sum_{q \leq X^{1/2}/2}^* \int_{|\beta| \leq \frac{1}{2qX^{1/2}}}^* q^{-4} \left(\sum_{\substack{a=1 \\ (a,q)=1}}^q |S_2(q, a)|^4 e(-ha/q) \right) |v_2(\beta)|^4 e(-h\beta) d\beta, \end{aligned}$$

where the notations \sum^* and \int^* mean either $q > Y$ or $Xq|\beta| > Y$. Whenever $(a, q) = 1$, one has by Lemma 2.2 that

$$|S_2(q, a)| = |S_2(q, 1)| \leq (2q)^{1/2}.$$

We obtain

$$\begin{aligned} & \left| \sum_{\substack{a=1 \\ (a,q)=1}}^q |S_2(q, a)|^4 e(-ha/q) \right| = |S_2(q, 1)|^4 \left| \sum_{\substack{a=1 \\ (a,q)=1}}^q e(-ha/q) \right| \\ & \leq 4q^2 \left| \sum_{\substack{a=1 \\ (a,q)=1}}^q e(-ha/q) \right| \leq 4q^2(q, h), \end{aligned}$$

whence

$$\int_{\mathfrak{m}_2} |f_2^*(\alpha)^4| e(-h\alpha) d\alpha \ll P_2^4 \sum_{q \leq X^{1/2}/2}^* \int_{|\beta| \leq \frac{1}{2qX^{1/2}}}^* \frac{q^{-2}(q, h)}{(1 + X|\beta|)^2} d\beta.$$

When $h = 0$, we have

$$\begin{aligned} \int_{\mathfrak{m}_2} |f_2^*(\alpha)^4| e(-h\alpha) d\alpha &\ll P_2^4 \sum_{q \leq X^{1/2}/2} \int_{|\beta| \leq \frac{1}{2qX^{1/2}}} q^{-1} (1 + X|\beta|)^{-2} d\beta \\ &\ll P_2^4 X^{-1} \log X. \end{aligned}$$

When $h \neq 0$, we get

$$\begin{aligned} \int_{\mathfrak{m}_2} |f_2^*(\alpha)^4| e(-h\alpha) d\alpha &\ll P_2^4 Y^{-1} \sum_{q \leq X^{1/2}/2} \int_{|\beta| \leq \frac{1}{2qX^{1/2}}} \frac{q^{-1}(q, h)}{1 + X|\beta|} d\beta \\ &\ll P_2^4 Y^{-1} X^{-1} (\log X) \sum_{q \leq X^{1/2}/2} q^{-1}(q, h) \\ &\ll P_2^4 Y^{-1} X^{-1+\varepsilon}. \end{aligned}$$

The conclusion (3.5) is established. Now we are able to estimate \mathcal{J} . When $s = 4$,

$$\mathcal{J} = \sum_{\substack{1 \leq x_1, x_2 \leq P_4 \\ n_1, n_2 \in \mathcal{Z}}} \eta(n_1) \overline{\eta(n_2)} \int_{\mathfrak{m}_2} |f_2^*(\alpha)^4| e(-(x_1^4 - x_2^4 + n_1 - n_2)\alpha) d\alpha.$$

On applying (3.5), we can deduce that

$$\begin{aligned} \mathcal{J} &\ll \sum_{\substack{1 \leq x_1, x_2 \leq P_4, \ n_1, n_2 \in \mathcal{Z} \\ x_1^4 - x_2^4 + n_1 - n_2 \neq 0}} P_4^{4+\varepsilon} Y^{-1} + \sum_{\substack{1 \leq x_1, x_2 \leq P_4, \ n_1, n_2 \in \mathcal{Z} \\ x_1^4 - x_2^4 + n_1 - n_2 = 0}} P_4^{4+\varepsilon} \\ &\ll P_4^{6+\varepsilon} Z^2 Y^{-1} + P_4^{4+\varepsilon} Z^2 + P_4^{5+\varepsilon} Z. \end{aligned}$$

Substituting $Y = P_4^{3/2+\tau} \psi(X)^2$, we finally obtain

$$\mathcal{J} \ll P_4^{4+1/2-\tau+\varepsilon} \psi(X)^{-2} Z^2 + P_4^{5+\varepsilon} Z,$$

whence

$$\mathcal{J}_2 \ll P_4^{4-\tau/2+\varepsilon} \psi(X)^{-1} Z + P_4^{4+1/4+\varepsilon} Z^{1/2}.$$

Similarly, when $s = 3$, one has

$$\mathcal{J} \ll P_4^{5/2-\tau+\varepsilon} \psi(X)^{-2} Z^2 + P_4^{4+\varepsilon} Z$$

whence

$$\mathcal{J}_2 \ll P_4^{3-\tau/2+\varepsilon} \psi(X)^{-1} Z + P_4^{4-1/4+\varepsilon} Z^{1/2}.$$

Therefore, we conclude that

$$(3.6) \quad \mathcal{J}_2 \ll P_4^{4-\frac{1}{4}+\frac{s-3}{2}+\varepsilon} Z^{1/2} + P_4^{s-\tau/2+\varepsilon} \psi(X)^{-1} Z.$$

Combining (3.3), (3.4) and (3.6), we conclude that

$$(3.7) \quad \mathcal{I}_2 \ll P_4^{4-\frac{1}{4}+\frac{s-3}{2}+\varepsilon} Z^{1/2} + P_4^{s-\tau/2+\varepsilon} \psi(X)^{-1} Z.$$

We complete the proof. \square

Lemma 3.3. *Let \mathcal{I}_3 be defined in (3.1). Then we have*

$$\mathcal{I}_3 \ll P_4^{4-\frac{1}{4}+\frac{s-3}{2}+\varepsilon} Z^{1/2} + P_4^{s-\tau+\varepsilon} \psi(X)^{-1} Z.$$

Proof. Similarly to (3.3) and (3.4), we can derive that

$$(3.8) \quad \mathcal{I}_3 \ll P_4^{4-\frac{1}{4}+\frac{s-3}{2}+\varepsilon} Z^{1/2} + P_4^{s-\frac{1}{4}+\varepsilon} Z + \mathcal{K},$$

where

$$\mathcal{K} = \int_{\mathfrak{m}_3} |f_2^*(\alpha)^2 f_4(\alpha)^s \mathcal{E}(\alpha)| d\alpha.$$

One has

$$\begin{aligned} \mathcal{K} \leqslant \sup_{\alpha \in \mathfrak{m}_3} |f_4(\alpha)| & \left(\int_{\mathfrak{m}_3} |f_2^*(\alpha)^2 f_4(\alpha)^4| d\alpha \right)^{1/2} \\ & \times \left(\int_{\mathfrak{m}_3} |f_2^*(\alpha)^2 f_4(\alpha)^{2(s-3)} \mathcal{E}(\alpha)^2| d\alpha \right)^{1/2}. \end{aligned}$$

In view of (2.3) and (2.4), we have for $\alpha \in \mathfrak{m}_3$ that

$$f_4(\alpha) \ll P_4 q^{-1/4} (1 + X|\alpha - a/q|)^{-1/4} + Y^{1/2} X^\varepsilon \ll P_4^{3/4+\tau/2+\varepsilon} \psi(X).$$

Since $f_2^*(\alpha) - f_2(\alpha) \ll P_2^{1/2}$ for $\alpha \in \mathfrak{m}_3$, we easily deduce that

$$\begin{aligned} \int_{\mathfrak{m}_3} |f_2^*(\alpha)^2 f_4(\alpha)^4| d\alpha & \ll P_2^{1/2} \int_0^1 |f_2(\alpha) f_4(\alpha)^4| d\alpha + \int_0^1 |f_2(\alpha)^2 f_4(\alpha)^4| d\alpha \\ & \ll P_4^{4+\varepsilon}. \end{aligned}$$

Therefore we arrive at

$$\mathcal{K} \ll P_4^{11/4+\tau/2+\varepsilon} \psi(X) \left(\int_{\mathfrak{m}_3} |f_2^*(\alpha)^2 f_4(\alpha)^{2(s-3)} \mathcal{E}(\alpha)^2| d\alpha \right)^{1/2}.$$

Similarly to (3.5), we have the following estimate

$$(3.9) \quad \int_{\mathfrak{M}(Y)} |f_2^*(\alpha)^2| e(-h\alpha) d\alpha = \begin{cases} O(P_4^\varepsilon), & \text{when } 0 < |h| \leqslant 2X, \\ O(P_4^\varepsilon Y), & \text{when } h = 0. \end{cases}$$

Note that

$$\begin{aligned} & \int_{\mathfrak{M}(Y)} |f_2^*(\alpha)^2| e(-h\alpha) d\alpha \\ &= \sum_{q \leqslant Y} \int_{|\beta| \leqslant \frac{Y}{qX}} q^{-2} \left(\sum_{\substack{a=1 \\ (a,q)=1}}^q |S_2(q, a)|^2 e(-ha/q) \right) |v_2(\beta)|^2 e(-h\beta) d\beta \\ &\ll P_2^2 \sum_{q \leqslant Y} \int_{|\beta| \leqslant \frac{Y}{qX}} q^{-1}(q, h) (1 + X|\beta|)^{-1} d\beta \\ &\ll (\log X) \sum_{q \leqslant Y} q^{-1}(q, h). \end{aligned}$$

The desired estimate (3.9) follows easily from above. For $s = 4$, we derive that

$$\begin{aligned}
& \int_{\mathfrak{m}_3} |f_2^*(\alpha)^2 f_4(\alpha)^2 \mathcal{E}(\alpha)^2| d\alpha \\
& \leq \int_{\mathfrak{M}(Y)} |f_2^*(\alpha)^2 f_4(\alpha)^2 \mathcal{E}(\alpha)^2| d\alpha \\
& = \sum_{\substack{n_1, n_2 \in \mathcal{Z} \\ 1 \leq x_1, x_2 \leq P_4}} \eta(n_1) \overline{\eta(n_2)} \int_{\mathfrak{M}(Y)} |f_2^*(\alpha)^2| e(-(n_1 - n_2 + x_1^4 - x_2^4)\alpha) d\alpha \\
& \ll P_4^{2+\varepsilon} Z^2 + P_4^\varepsilon Y (P_4^\varepsilon Z^2 + P_4 Z) \\
& \ll (P_4^{2+\varepsilon} + P_4^{3/2+\tau+\varepsilon} \psi(X)^2) Z^2 + P_4^{5/2+\tau+\varepsilon} \psi(X)^2 Z,
\end{aligned}$$

whence

$$\mathcal{K} \ll (P_4^{15/4+\tau/2+\varepsilon} \psi(X) + P_4^{7/2+\tau+\varepsilon} \psi(X)^2) Z + P_4^{4+\tau+\varepsilon} \psi(X)^2 Z^{1/2}.$$

In particular, we have

$$\mathcal{K} \ll P_4^{4+\frac{1}{4}+\varepsilon} Z^{1/2} + P_4^{4-\tau+\varepsilon} \psi(X)^{-1} Z$$

provided that $\psi(X) \ll X^{1/64-\tau}$. For $s = 3$, by (3.9) we have

$$\int_{\mathfrak{m}_3} |f_2^*(\alpha)^2 \mathcal{E}(\alpha)^2| d\alpha \ll P_4^\varepsilon Z^2 + P_4^{3/2+\tau+\varepsilon} \psi(X)^2 Z,$$

whence

$$\mathcal{K} \ll P_4^{11/4+\tau/2+\varepsilon} \psi(X) Z + P_4^{7/2+\tau+\varepsilon} \psi(X)^2 Z^{1/2}.$$

When $\psi(X) \ll X^{1/64-\tau}$, one has

$$\mathcal{K} \ll P_4^{4-\frac{1}{4}+\varepsilon} Z^{1/2} + P_4^{3-\tau+\varepsilon} \psi(X)^{-1} Z.$$

We conclude from above that

$$(3.10) \quad \mathcal{K} \ll P_4^{4-\frac{1}{4}+\frac{s-3}{2}+\varepsilon} Z^{1/2} + P_4^{s-\tau+\varepsilon} \psi(X)^{-1} Z.$$

By (3.8) and (3.10), we obtain

$$(3.11) \quad \mathcal{I}_3 \ll P_4^{4-\frac{1}{4}+\frac{s-3}{2}+\varepsilon} Z^{1/2} + P_4^{s-\tau+\varepsilon} \psi(X)^{-1} Z.$$

We complete the proof. □

Lemma 3.4. *Let \mathcal{I}_4 be defined in (3.1). Then we have*

$$\mathcal{I}_4 \ll Z P_4^{s-(s-2)\nu/4+\varepsilon}.$$

Proof. In view of (2.3) and (2.4), for $\alpha \in \mathfrak{M}_{P_4}(q, a)$, one has

$$\begin{aligned}
f_4(\alpha) & \ll P_4 w_4(q) (1 + X|\alpha - a/q|)^{-1/4} + P_4^{1/2+\varepsilon} \\
& \ll P_4^{1+\varepsilon} w_4(q) (1 + X|\alpha - a/q|)^{-1/4},
\end{aligned}$$

and

$$f_2(\alpha) \ll P_2 q^{-1/2} (1 + X|\alpha - a/q|)^{-1/2}.$$

Therefore we obtain

$$\begin{aligned} \mathcal{I}_4 &\ll Z \sup_{\alpha \in \mathfrak{m}_4} |f_4(\alpha)|^{s-2} \int_{\mathfrak{M}(P_4)} |f_4(\alpha) f_2(\alpha)|^2 d\alpha \\ &\ll Z P_4^{(s-2)(1-\nu/4)+\varepsilon} P_4^2 P_2^2 \sum_{q \leq P_4} w_4(q)^2 \int_{|\beta| \leq \frac{P_4}{qX}} (1 + X|\beta|)^{-3/2} d\beta \\ &\ll Z P_4^{2+(s-2)(1-\nu/4)+\varepsilon} \sum_{q \leq P_4} w_4(q)^2. \end{aligned}$$

In light of Lemma 2.4 by Kawada and Wooley [7], one can conclude that

$$(3.12) \quad \mathcal{I}_4 \ll Z P_4^{2+(s-2)(1-\nu/4)+\varepsilon} \ll Z P_4^{s-(s-2)\nu/4+\varepsilon}.$$

The desired estimate is established. \square

Proof of Theorem 1.1. We denote by $Z_s(X)$ the set of integers n with $X/2 < n \leq X$ for which the lower bound

$$|R_s(n) - c_s \Gamma(\frac{5}{4})^4 \mathfrak{S}_s(n) n^{s/4}| > n^{s/4} \psi(n)^{-1}$$

holds, and we abbreviate $\text{card}(Z_s(X))$ to Z_s . It follows from (2.1) and Lemma 2.1 that for $n \in Z_s(X)$,

$$\left| \int_{\mathfrak{m}} f_2(\alpha)^2 f_4(\alpha)^s e(-n\alpha) d\alpha \right| \gg X^{s/4} \psi(X)^{-1},$$

whence

$$\sum_{n \in Z_s(X)} \left| \int_{\mathfrak{m}} f_2(\alpha)^2 f_4(\alpha)^s e(-n\alpha) d\alpha \right| \gg Z_s X^{s/4} \psi(X)^{-1}.$$

We choose complex numbers $\eta(n)$, with $|\eta(n)| = 1$, satisfying

$$\left| \int_{\mathfrak{m}} f_2(\alpha)^2 f_4(\alpha)^s e(-n\alpha) d\alpha \right| = \eta(n) \int_{\mathfrak{m}} f_2(\alpha)^2 f_4(\alpha)^s e(-n\alpha) d\alpha.$$

Then we define the exponential sum $\mathcal{E}_s(\alpha)$ by

$$\mathcal{E}_s(\alpha) = \sum_{n \in Z_s(X)} \eta(n) e(-n\alpha).$$

One finds that

$$(3.13) \quad Z_s X^{s/4} \psi(X)^{-1} \ll \int_{\mathfrak{m}} |f_2(\alpha)^2 f_4(\alpha)^s \mathcal{E}_s(\alpha)| d\alpha.$$

Note that $\mathfrak{m} = \mathfrak{m}_1 \cup \mathfrak{m}_2 \cup \mathfrak{m}_3 \cup \mathfrak{m}_4$. Now we conclude from Lemmata 3.1-3.4 and (3.13) that

$$Z_s X^{s/4} \psi(X)^{-1} \ll P_4^{4-\frac{1}{4}+\frac{s-3}{2}+\varepsilon} Z_s^{1/2} + P_4^{s-\delta} \psi(X)^{-1} Z_s$$

for some sufficiently small positive number δ . Therefore we have

$$(3.14) \quad Z_s X^{s/4} \psi(X)^{-1} \ll X^{1-\frac{1}{16}+\frac{s-3}{8}+\varepsilon} Z_s^{1/2}.$$

The estimate (3.14) implies $Z_3 \ll X^{3/8+\varepsilon} \psi(X)^2$ and $Z_4 \ll X^{1/8+\varepsilon} \psi(X)^2$. The proof of Theorem 1.1 is completed by summing over dyadic intervals. \square

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